ASYNCHRONOUS PROBABILISTIC COUPLINGS

in Higher-Order Separation Logic

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Motivating example

$\text{let } b = \text{flip in}$

$\lambda \_ .\, b$

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\begin{array}{l}
\textsf{let } r = \textsf{ref}(\textsf{None}) \textsf{ in} \\
\lambda\_.\ \textsf{match } !r \textsf{ with} \\
\quad\ \ \textsf{Some}(b) \Rightarrow b \\
\quad | \ \textsf{None} \quad\ \Rightarrow \textsf{let } b = \textsf{flip in} \\
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While this example seems esoteric, the pattern shows up in many places: cryptographic security, hash functions, lazily-sampled big integers, ...

The pRHL approach

For such properties, people have developed probabilistic relational Hoare logics, where sampling statements are related through so-called coupling rules.

pRHL-couple

$$\frac{}{\{\text{True}\}\ \text{flip} \sim \text{flip}\ \{v_1, v_2.\ \exists(b : \mathbb{B}).\ b = v_1 = v_2\}}$$

However, it requires you to "synchronize" the probabilistic choices.

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This work

Goal: prove contextual equivalence of

… probabilistic programs written in an expressive programming language
… using a higher-order separation logic, called Clutch,
… and asynchronous probabilistic couplings

while mechanizing everything in the Coq proof assistant.

The $\mathbf{F}_{\mu,\mathsf{ref}}^{\mathsf{rand}}$ language

An ML-like language with higher-order (recursive) functions, higher-order state, impredicative polymorphism, ..., and probabilistic uniform sampling.

$$
\begin{aligned}
e \in \mathsf{Expr} &::= \ \ldots \ \mid \ \mathsf{rand}(e) \\
\tau \in \mathsf{Type} &::= \alpha \mid \mathsf{unit} \mid \mathsf{bool} \mid \mathsf{int} \mid \tau \times \tau \mid \tau + \tau \mid \tau \to \tau \mid \\
&\quad\ \forall \alpha.\, \tau \mid \exists \alpha.\, \tau \mid \mu\, \alpha.\, \tau \mid \mathsf{ref}\, \tau
\end{aligned}
$$

and a standard typing judgment $\Gamma \vdash e : \tau$.

Operational semantics

$$(\lambda x.\, e_1)e_2 \to^1 e_1[e_2/x]$$

$$\vdots$$

$$\mathsf{rand}(N) \to^{1/(N+1)} n \qquad\qquad n \in \{0, 1, \ldots, N\}$$

For this presentation, we only consider flip $\triangleq$ rand(1).

Contextual refinement

The property of interest is contextual refinement.

$$\Gamma \vdash e_1 \precsim_{\mathsf{ctx}} e_2 : \tau \quad \triangleq \quad \forall \tau', (\mathcal{C} : (\Gamma \vdash \tau) \Rightarrow (\emptyset \vdash \tau')), \sigma. \\ \mathrm{term}(\mathcal{C}[\, e_1 \,], \sigma) \leq \mathrm{term}(\mathcal{C}[\, e_2 \,], \sigma)$$

and $\Gamma \vdash e_1 \simeq_{\mathsf{ctx}} e_2 : \tau$ follows as refinement in both directions.

Proving contextual refinement

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2. A logical refinement judgment (a “logical” logical relation)

$$\Gamma \vDash e_1 \precsim e_2 : \tau$$

that implies contextual refinement.

Refinement judgment

The judgment

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should be read as “in env. Γ, expression e_1 refines expression e_2 at type τ”.

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Theorem (Fundamental)

If $\Gamma \vdash e : \tau$ then $\Gamma \vDash e \precsim e : \tau$.

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Theorem (Fundamental)

If $\Gamma \vdash e : \tau$ then $\Gamma \vDash e \precsim e : \tau$.

Theorem (Soundness)

If $\Gamma \vDash e_1 \precsim e_2 : \tau$ then $\Gamma \vdash e_1 \precsim_{\mathsf{ctx}} e_2 : \tau$.

Symbolic execution rules

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$$\frac{e_1 \overset{\text{pure}}{\rightsquigarrow} e_1' \qquad \Gamma \vDash K[\,e_1'\,] \precsim e_2 : \tau}{\Gamma \vDash K[\,e_1\,] \precsim e_2 : \tau}$$

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Relational separation logic

Clutch is built on top of the Iris separation logic framework

$$
\begin{aligned}
P, Q \in \textsf{iProp} ::= {} & \textsf{True} \mid \textsf{False} \mid P \wedge Q \mid P \vee Q \mid P \Rightarrow Q \mid && \text{(propositional)} \\
& \forall x.\, P \mid \exists x.\, P \mid && \text{(higher-order)} \\
& P * Q \mid P \mathrel{-\!\!*} Q \mid \ell \mapsto v \mid && \text{(separation)} \\
& \square\, P \mid \triangleright P \mid \fbox{a} \mid \boxed{P} \mid \ldots \mid && \text{(Iris)} \\
& \textsf{wp}\ e\ \{\Phi\} \mid \textsf{spec}(e) \mid \ldots && \text{(Clutch)}
\end{aligned}
$$

within which we derive $\Gamma \vDash e_1 \precsim e_2 : \tau$.

- The connectives wp $e\,\{\Phi\}$ and spec(e) form a coupling logic.

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- Soundness of the coupling logic will allow us to conclude that there exists a probabilistic coupling of the execution of e_1 and e_2.

Asynchronous couplings

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Operationally, we extend the execution state with dynamically allocated tapes.

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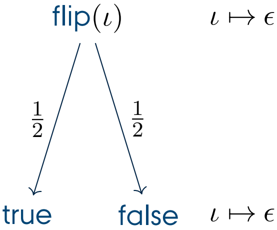
Presampling tapes

$\mathsf{flip}(\iota)$ $\qquad$ $\iota \mapsto \epsilon$

Presampling tapes



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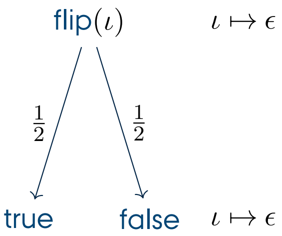

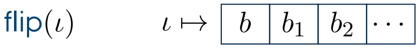

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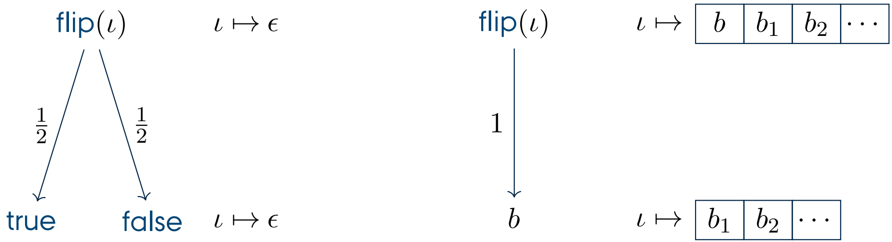

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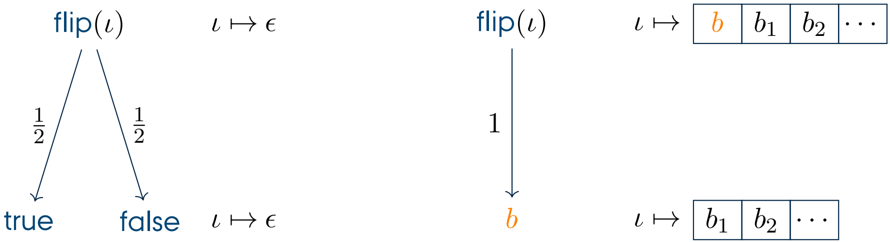

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$$\iota : \mathsf{tape} \vdash \mathsf{flip} \simeq_{\mathsf{ctx}} \mathsf{flip}(\iota) : \mathsf{bool}$$

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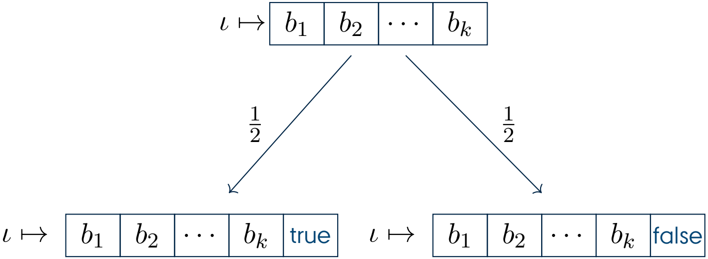

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$$\iota \hookrightarrow \vec{b}$$

that denotes ownership of a tape ι and its contents $\vec{b}$.

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It—locally—turns reasoning about probabilistic choice into reasoning about state.

Asynchronous couplings

With presampling tapes, we can synchronously couple tape samplings with program samplings

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$$e$$

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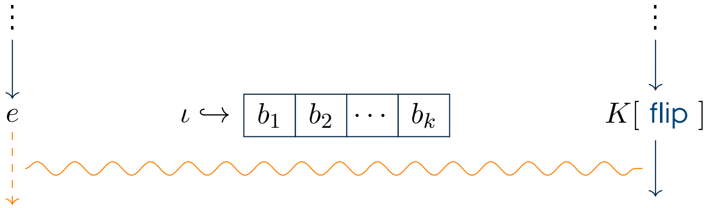
$$K[\ \mathsf{flip}\ ]$$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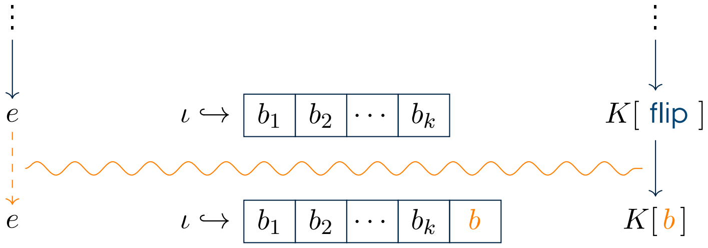

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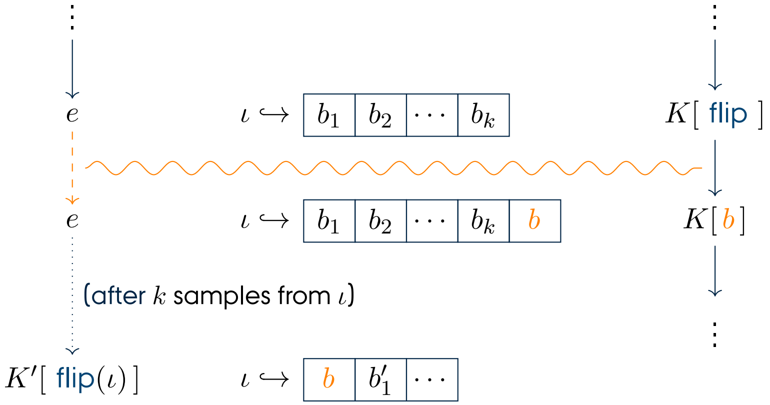
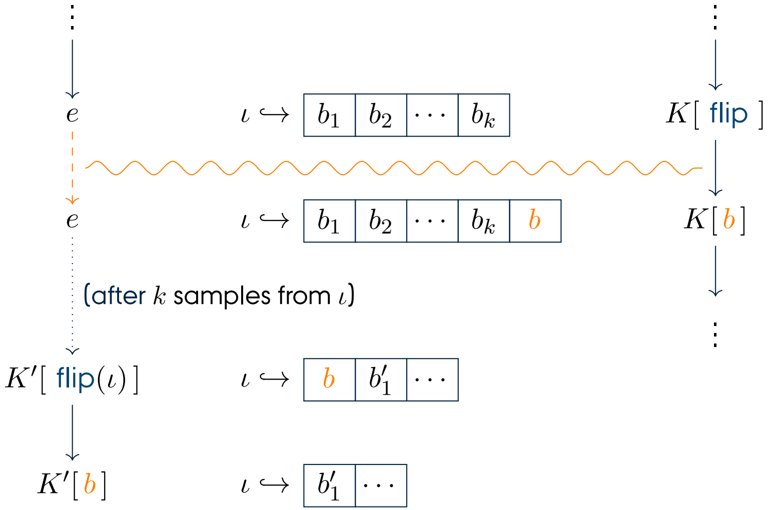


e
$\iota \hookrightarrow$ b_1 b_2 $\cdots$ b_k
$K[$ flip $]$
e
$\iota \hookrightarrow$ b_1 b_2 $\cdots$ b_k b
$K[b]$
(after k samples from ι)
$K'[$ flip(ι) $]$
$\iota \hookrightarrow$ b b'_1 $\cdots$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e
$\iota \hookrightarrow$ b_1 b_2 $\cdots$ b_k b
$K[\ b\ ]$
(after k samples from ι)
$K'[\ \text{flip}(\iota)\ ]$
$\iota \hookrightarrow$ b b'_1 $\cdots$
$K'[\ b\ ]$
$\iota \hookrightarrow$ b'_1 $\cdots$

Motivating example cont'd

$$\begin{aligned}
&\textsf{let } r = \textsf{ref}(\textsf{None}) \textsf{ in} \\
&\lambda_.\ \textsf{match } !r \textsf{ with} \\
&\quad \textsf{Some}(b) \Rightarrow b \\
&\quad \mid \textsf{None} \Rightarrow \textsf{let } b = \textsf{flip in} \\
&\qquad\qquad r \leftarrow \textsf{Some}(b); \\
&\qquad\qquad b \\
&\quad \textsf{end}
\end{aligned}
\quad \precsim_{\textsf{ctx}} \quad
\begin{aligned}
&\textsf{let } b = \textsf{flip in} \\
&\lambda_.\ b
\end{aligned}$$

Motivating example cont'd

$$\begin{array}{l}
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda_.\ \mathsf{match}\ !r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\qquad\qquad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ \iota = \mathsf{tape}\ \mathsf{in} \\
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda_.\ \mathsf{match}\ !r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}(\iota)\ \mathsf{in} \\
\qquad\qquad r \leftarrow \mathsf{Some}(b); \\
\qquad\qquad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\lambda_.\ b
\end{array}$$

Motivating example cont'd

$$
\begin{array}{l}
\text{let } r = \text{ref(None) in} \\
\lambda\_.\ \text{match } !r \text{ with} \\
\quad \text{Some}(b) \Rightarrow b \\
\quad \mid \text{None} \Rightarrow \text{let } b = \text{flip in} \\
\qquad\qquad r \leftarrow \text{Some}(b); \\
\qquad\qquad b \\
\quad \text{end}
\end{array}
\precsim_{\text{ctx}}
\begin{array}{l}
\text{let } \iota = \text{tape in} \\
\text{let } r = \text{ref(None) in} \\
\lambda\_.\ \text{match } !r \text{ with} \\
\quad \text{Some}(b) \Rightarrow b \\
\quad \mid \text{None} \Rightarrow \text{let } b = \text{flip}(\iota) \text{ in} \\
\qquad\qquad r \leftarrow \text{Some}(b); \\
\qquad\qquad b \\
\quad \text{end}
\end{array}
\precsim_{\text{ctx}}
\begin{array}{l}
\text{let } b = \text{flip in} \\
\lambda\_.\ b
\end{array}
$$

Motivating example cont'd

$$
\begin{array}{l}
\textsf{let}\ r = \textsf{ref}(\textsf{None})\ \textsf{in} \\
\lambda\_.\ \textsf{match}\ !r\ \textsf{with} \\
\quad \textsf{Some}(b) \Rightarrow b \\
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\textsf{end}
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\textsf{let}\ \iota = \textsf{tape}\ \textsf{in} \\
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\qquad\qquad\qquad b \\
\textsf{end}
\end{array}
\quad \precsim_{\textsf{ctx}} \quad
\begin{array}{l}
\textsf{let}\ b = \textsf{flip}\ \textsf{in} \\
\lambda\_.\ b
\end{array}
$$

Motivating example cont'd

$$
\begin{array}{l}
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda\_.\ \mathsf{match}\ !r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
\quad \mid \mathsf{None} \Rightarrow \mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
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\qquad\qquad b \\
\quad \mathsf{end}
\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ \iota = \mathsf{tape}\ \mathsf{in} \\
\mathsf{let}\ r = \mathsf{ref}(\mathsf{None})\ \mathsf{in} \\
\lambda\_.\ \mathsf{match}\ !r\ \mathsf{with} \\
\quad \mathsf{Some}(b) \Rightarrow b \\
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\end{array}
\precsim_{\mathsf{ctx}}
\begin{array}{l}
\mathsf{let}\ b = \mathsf{flip}\ \mathsf{in} \\
\lambda\_.\ b
\end{array}
$$

Summary

- A higher-order probabilistic relational separation logic, Clutch, for proving contextual refinement of higher-order probabilistic programs.
- A proof method for asynchronous couplings.
- Many more examples in the paper:
  → Cryptographic security, hash functions, lazily-sampled big integers, ...
- Full mechanization of all results in the Coq proof assistant.

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Paper `https://doi.org/10.1145/3632868`
Coq dev. `https://github.com/logsem/clutch`

Extras

Let $\mathrm{step}(\rho) \in \mathcal{D}(\mathsf{Cfg})$ be the distribution of single step reduction of $\rho \in \mathsf{Cfg}$.

$$\mathrm{exec}_n(e, \sigma) \triangleq \begin{cases} \mathbf{0} & \text{if } e \notin \mathsf{Val} \text{ and } n = 0 \\ \mathrm{ret}(e) & \text{if } e \in \mathsf{Val} \\ \mathrm{step}(e, \sigma) \gg\!= \mathrm{exec}_{(n-1)} & \text{otherwise} \end{cases}$$

$$\mathrm{exec}(\rho)(v) \triangleq \lim_{n \to \infty} \mathrm{exec}_n(\rho)(v)$$

$$\mathrm{term}(\rho) \triangleq \sum_{v \in \mathsf{Val}} \mathrm{exec}(\rho)(v)$$

Definition (Coupling)

Let $\mu_1 \in \mathcal{D}(A)$, $\mu_2 \in \mathcal{D}(B)$. A sub-distribution $\mu \in \mathcal{D}(A \times B)$ is a coupling of μ_1 and μ_2 if

1. $\forall a.\ \sum_{b \in B} \mu(a, b) = \mu_1(a)$
2. $\forall b.\ \sum_{a \in A} \mu(a, b) = \mu_2(b)$

Given a relation $R \subseteq A \times B$ we say μ is an R-coupling if furthermore $\mathrm{supp}(\mu) \subseteq R$. We write $\mu_1 \sim \mu_2 : R$ if there exists an R-coupling of μ_1 and μ_2.

Lemma

If $\mu_1 \sim \mu_2 : (=)$ then $\mu_1 = \mu_2$.

Definition (Left-Partial Coupling)

Let $\mu_1 \in \mathcal{D}(A), \mu_2 \in \mathcal{D}(B)$. A sub-distribution $\mu \in \mathcal{D}(A \times B)$ is a left-partial coupling of μ_1 and μ_2 if

1. $\forall a.\ \sum_{b \in B} \mu(a, b) = \mu_1(a)$
2. $\forall b.\ \sum_{a \in A} \mu(a, b) \leq \mu_2(b)$

Given a relation $R \subseteq A \times B$ we say μ is an R-left-partial-coupling if furthermore $\mathrm{supp}(\mu) \subseteq R$. We write $\mu_1 \lesssim \mu_2 : R$ if there exists an R-left-partial-coupling of μ_1 and μ_2.

Lemma

If $\mu_1 \sim \mu_2 : R$ then $\mu_1 \lesssim \mu_2 : R$.

Lemma

If $\mu_1 \lesssim \mu_2 : (=)$ then $\forall a.\ \mu_1(a) \leq \mu_2(a)$.

The adequacy theorem relies on the fact that presampling does not matter.

Lemma (Erasure)

If $\sigma_1(\iota) \in \mathsf{dom}(\sigma_1)$ then

$$\mathrm{exec}_n(e_1, \sigma_1) \sim (\mathrm{step}_\iota(\sigma_1) \gg\!= \lambda\sigma_2.\ \mathrm{exec}_n(e_1, \sigma_2)) : (=)$$